# Exact analytical solutions for the variational equations derived from the nonlinear Schrödinger equation

A. B. Moubissi*
*Département de Physique de l'Université des Sciences et Techniques de Masuku, B.P. 943 Franceville, Gabon*

K. Nakkeeran† and Abdosllam M. Abobaker
*School of Engineering, Fraser Noble Building, King's College, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom*
(Received 3 May 2007; published 8 August 2007)

By means of the variational formalism for the nonlinear Schrödinger equation, we find an explicit relation for the power of a pulse in terms of its duration, chirp and fiber parameters (group-velocity dispersion and self-phase modulation parameters). Then, using that relation, we derive the explicit analytical expressions for the variational equations corresponding to the amplitude, width, and chirp of the pulse. The derivation of the analytical expressions for the variational equations is possible for the condition when the Hamiltonian of the system is zero. Finally, for Gaussian and hyperbolic secant ansatz, we show good agreement between the results obtained from the analytical expressions and the direct numerical simulation of the nonlinear Schrödinger equation.

DOI: 10.1103/PhysRevE.76.026603 PACS number(s): 05.45.Yv, 42.81.Dp, 42.65.Tg

## I. INTRODUCTION

Almost all the complex nonlinear partial differential equations governing the nonlinear systems are in the family of nonlinear Schrödinger equation (NLSE). Many important physical systems like, nonlinear fiber optics, Bose-Einstein condensate (BEC), water waves, plasma waves, etc., are governed by the NLSE [1]. The propagation of intense optical pulse in a nonlinear Kerr medium has attracted considerable attention from the scientific community. In this case as well the dynamics of the nonlinear pulse propagation is governed by the NLSE, in which the group-velocity dispersion (GVD) and self-phase modulation (SPM), form a basic set of optical processes describing a broad range of realistic physical situations [2]. Other equations in the family of NLSE are usually called as higher-order NLSE or generalized NLSE, which may include other effects like optical losses, high-order dispersion, stimulated inelastic scattering and self-steepening effects [2]. The NLSE is studied extensively in order to understand the influence of combining those effects. A particular initial condition of the pulse leads to a particular dynamical process during its propagation in the optical fiber. The more famous of such a process is the conventional soliton. It can be observed when the effect of anomalous GVD is exactly balanced by the SPM in optical fibers [3]. Thus formed (soliton) pulse can then propagate without any deformation of its shape.

Under special cases, the NLSE is completely integrable and the corresponding soliton solutions can be derived using the standard technique called inverse scattering transform [4]. But the family of NLSE equations governing most practical cases like conventional fiber transmission system, dispersion-managed (DM) fiber system are not completely integrable in general. Even though some perturbation methods were reported to investigate the behavior of physically interesting nonintegrable NLSE family, researchers working in nonlinear optics and other fields mostly rely on numerical methods and Lagrangian variational method to study the system dynamics [5,6]. Variational method is one of the widely used approximation techniques which has been applied to study the dynamics of various pulse parameters with respect to the system parameters, to estimate pulse interaction length and to find the fixed point solutions of DM fiber systems [6]. There are numerous works related to the modification of the variational method to include various important effects neglected in the formalism of typical variational analysis. One of the important factors considered by various researcher to modify the variational method has been on the radiation induced solitons interactions [7–9]. Kuznetsov *et al.*, considered the nonlinear interaction of solitons and radiations [10], where they reported the unsuccessful application of the variational method. Nevertheless all these works were based on the fundamental variational method and its success or failure. Mikhailov has reported an interesting article about the validity of the variational analysis [11].

In this work, we derive the exact analytical expressions for the variational equations corresponding to the amplitude, width and chirp under the condition when the Hamiltonian of the system is zero. We finally show that the results obtained from the solutions of the variational equations are in good agreement with the results obtained from the direct numerical simulation of the NLSE. This paper is organized as follows. In Sec. II, we present the variational analysis of the NLSE. The exact analytical expressions for the amplitude, width and chirp is derived in Sec. III. The comparison between the results obtained by analytics and numerics is presented in Sec. IV. Finally, in Sec. V, we conclude.

## II. VARIATIONAL ANALYSIS

The propagation of nonlinear pulses in optical medium, like fibers, obeys the NLSE [2–4]

*moubissi@hotmail.com
†K.Nakkeeran@abdn.ac.uk

 

$$\psi_z + \frac{i\beta}{2}\psi_{tt} - i\gamma|\psi|^2\psi = 0, \tag{1}$$

where $\psi$ is the slowly varying envelope of the axial electrical field, which is a function of time $t$ and space $z$. $\beta$ and $\gamma$ represent the GVD and SPM parameters, respectively. We have reported a collective variable method to analyze the nonlinear pulse propagation in fiber system in terms of pulse parameters like the amplitude, temporal position, pulse width, chirp, frequency, and phase [12,13]. The above parameters are called the collective variables. Using the collective variable method and assuming the following ansatz for $\psi$ in the NLSE (1)

$$\psi = x_1\phi\left(\frac{t-x_2}{x_3}\right)\exp\left[\frac{ix_4}{2}(t-x_2)^2 + ix_5(t-x_2) + ix_6\right], \tag{2}$$

the pulse width evolution equation can be derived as (overhead dot represents the derivative with respect to $z$) [14]

$$\ddot{x}_3 = \frac{\alpha_1\beta^2}{x_3^3} + \frac{\alpha_2\beta\gamma E_0}{x_3^2}. \tag{3}$$

Here $x_1, x_2, \sqrt{2\ln 2}x_3$, $x_4/(2\pi), x_5/(2\pi)$ and $x_6$ represent the pulse amplitude, temporal position, pulse width (FWHM), chirp, velocity and phase, respectively. $\alpha_1$ and $\alpha_2$ are two constants which depend on the particular choice of the analytical function $\phi$ assumed to represent the pulse profile. $E_0=x_1(0)^2x_3(0)=x_1(z)^2x_3(z)$ is a constant related to the energy of the pulse. On integrating Eq. (3) yields

$$\frac{\dot{x}_3^2}{2} = -\frac{\alpha_1\beta^2}{2x_3^2} - \frac{\alpha_2\beta\gamma E_0}{x_3} + \mathcal{C}, \tag{4}$$

where $\mathcal{C}$ is the integration constant related to the Hamiltonian of the system through the relation [15,16]

$$\mathcal{H} = \frac{\mathcal{C}}{\beta} \tag{5}$$

and is given by [14,16]

$$\mathcal{C} = \frac{\alpha_1\beta^2}{2x_{30}^2} + \frac{\alpha_2\beta\gamma E_0}{x_{30}}. \tag{6}$$

$x_{30}$ is the initial pulse width at $z=z_0$. Equation (6) shows that the energy of a pulse while propagating in a dispersive and nonlinear medium, like an optical fibre, depends directly on the parameter $\mathcal{C}$. In other words, a given value of $\mathcal{C}$ will lead to a particular solution for Eq. (4). From Eq. (6), it is clear that the constant of integration $\mathcal{C}$ can take any real positive, negative or zero values [16]. The solution of Eq. (4) thus depends on the value of the parameter $\mathcal{C}$. When $\mathcal{C}$ is nonzero then the solution of Eq. (4) becomes transcendental and gives the length of the pulse propagation with respect to the assumed initial and final values for the pulse width [see Eq. (9) in Ref. [14]]. This solution of Eq. (4) with nonzero value for $\mathcal{C}$ has been successfully exploited for the analytical design of DM fiber transmission system for a given set of pulse and fiber parameters [14,17–19].

## III. ANALYTICAL SOLUTIONS

In this work we are interested in the solution of the differential equation (4) when the constant of integration $\mathcal{C}$ is set to zero (system with zero value for the Hamiltonian). For this case we first find, according to Eq. (4), the energy $E_0$ is given by [using the variational equation of the pulse width as $\dot{x}_3=-\beta x_3(z)x_4(z)$ [14]]

$$E_0 = -\frac{\beta}{2\alpha_2\gamma}x_3^3(z)x_4^2(z) - \frac{\alpha_1\beta}{2\gamma\alpha_2}\frac{1}{x_3(z)}. \tag{7}$$

With $\mathcal{C}=0$ in Eq. (6), $E_0$ can also be expressed in terms of the initial pulse width as

$$E_0 = -\frac{\alpha_1}{2\alpha_2}\frac{\beta}{\gamma x_{30}}. \tag{8}$$

From both the expressions (7) and (8) it is clear that $E_0$ depends on the choice of the ansatz function $\phi$ representing the shape of the pulse. The integration of Eq. (4) with $\mathcal{C}=0$ yields

$$\frac{2}{3(2\alpha_2\gamma\beta E_0)^2}(-\alpha_1\beta^2 - 2\alpha_2\gamma\beta E_0x_3)^{3/2} - \frac{-2\alpha_1\beta^2}{(2\alpha_2\gamma\beta E_0)^2} \times(-\alpha_1\beta^2 - 2\alpha_2\gamma\beta E_0x_3)^{1/2} = \pm(z+\sigma), \tag{9}$$

where

$$\sigma = \frac{2}{3(2\alpha_2\gamma\beta E_0)^2}(-\alpha_1\beta^2 - 2\alpha_2\gamma\beta E_0x_{30})^{3/2} - \frac{-2\alpha_1\beta^2}{(2\alpha_2\gamma\beta E_0)^2}(-\alpha_1\beta^2 - 2\alpha_2\gamma\beta E_0x_{30})^{1/2} - z_0 \tag{10}$$

is the initial condition at the launch point $z_0$. Taking the factor $(-\alpha_1\beta^2-2\alpha_2\gamma\beta E_0x_3)^{1/2}$ in Eq. (9) yields

$$\frac{-2(-\alpha_1\beta^2 - 2\alpha_2\beta\gamma E_0x_3)(-2\alpha_1\beta^2 + 2\alpha_2\beta\gamma E_0x_3)}{3(2\alpha_2\beta\gamma E_0)^2} = \pm(z+\sigma). \tag{11}$$

Taking the square of Eq. (11) and rearranging we get

$$y^3 - 3\alpha_1\beta y^2 - 4\alpha^3\beta^3 + \frac{9\beta}{4}(-2\alpha_2\gamma E_0)^4(z+\sigma)^2 = 0, \tag{12}$$

where $y=2\alpha_2\gamma E_0x_3$. Then, replacing $y$ with $r+\alpha_1\beta$ we get the following cubic equation,

$$r^3 + ar + b = 0, \tag{13}$$

with $a=-3\alpha_1^2\beta^2$ and $b=2\alpha_1^3\beta^3+9\beta(-2\alpha_2\gamma E_0)^4(z+\sigma)^2/4$. The roots of Eq. (13) can be written as

$$r(\alpha_1,\alpha_2,z) = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3}. \tag{14}$$

Equation (14) has a physical constraint as $b^2/4+a^3/27$ to be positive. Finally, replacing $r$, $a$ and $b$ by their respective expression, we find that the pulse width is given by

$$x_3(z) = \frac{1}{\gamma E_0}[(P + \sqrt{P^2 + Q})^{1/3} + (P - \sqrt{P^2 + Q})^{1/3}] + \frac{1}{2}\alpha_1\alpha_2\beta\gamma E_0, \qquad (15)$$

where

$$P = -\frac{1}{2}\left[\frac{1}{4}\frac{\alpha_1^3}{\alpha_2^3} + \frac{9}{2}\alpha_2\beta\gamma^4 E_0^4 (z-\sigma)^2\right]$$

and

$$Q = \left(-\frac{\beta^2}{4}\frac{\alpha_1}{\alpha_2}\right)^3.$$

This relation describes explicitly the evolution of the pulse width, in a dispersive and nonlinear optical fiber medium, with respect to the GVD ($\beta$), SPM ($\gamma$), initial pulse width ($x_{30}$), energy ($E_0$) and the propagation distance ($z$). Using the expression (15) for $x_3$ in $x_1 = \sqrt{E_0/x_3}$ and $x_4 = -\dot{x}_3/(\beta x_3)$ [14], ($\dot{x}_3$ can be readily calculated from Eq. (4) with $\mathcal{C}=0$), we find the exact analytical expressions for the amplitude and chirp of the pulse as

$$x_1(z) = \frac{\sqrt{E_0}}{\sqrt{\frac{1}{\gamma E_0}[(P + \sqrt{P^2 + Q})^{1/3} + (P - \sqrt{P^2 + Q})^{1/3}] + \frac{1}{2}\alpha_1\alpha_2\beta\gamma E_0}}$$

and

$$x_4(z) = \pm\frac{\sqrt{\alpha 1\beta^2 - 2\alpha_2\beta[(P + \sqrt{P^2 + Q})^{1/3} + (P - \sqrt{P^2 + Q})^{1/3}] - \alpha_1(\alpha_2\beta\gamma E_0)^2}}{\beta\left\{\frac{1}{\gamma E_0}[(P + \sqrt{P^2 + Q})^{1/3} + (P - \sqrt{P^2 + Q})^{1/3}] + \frac{1}{2}\alpha_1\alpha_2\beta\gamma E_0\right\}^2}.$$

## IV. NUMERICAL SIMULATIONS

For different choice of the ansatz function $\phi$ in Eq. (2), the constants $\alpha_1$ and $\alpha_2$ can take different values. Most commonly used profile for the pulse propagation in optical fibers is Gaussian or hyperbolic secant ansätze. When we consider the Gaussian function for $\phi$ as

$$\phi\left(\frac{t-x_2}{x_3}\right) = \exp\left[\frac{(t-x_2)^2}{x_3^2}\right], \qquad (16)$$

then the constants $\alpha_1$ and $\alpha_2$ take the values 4 and $\sqrt{2}$, respectively. When we consider the hyperbolic secant function for $\phi$ as

$$\phi\left(\frac{t-x_2}{x_3}\right) = \operatorname{sech}\left(\frac{t-x_2}{x_3}\right), \qquad (17)$$

then the constants $\alpha_1$ and $\alpha_2$ take the same value as $4/\pi^2$.

For completeness we compare the results obtained using the analytical solutions of the variational equations and direct numerical simulation of the NLSE (1). For this we consider the pulse propagation in an optical fiber with GVD parameter $\beta = 1$ ps/(nm km), SPM parameter $\gamma = 0.0014$ m$^{-1}$ W$^{-1}$, initial pulse width $x_{30} = 3$ ps and length 20 km. Figure 1 shows the results obtained from the analytical expressions (solid curves) and numerics (dashed curves) for a Gaussian pulse propagation. Figure 2 shows the results obtained from the analytical expressions (solid curves) and numerics (dashed curves) for a hyperbolic secant pulse propagation. In both Figs. 1 and 2, we have also plotted the residual field energy (RFE) which is the difference between the energies of the pulse calculated from numerics and the best fit ansatz function [12,13]. The RFE can give an idea how good is the shape of the pulse maintained during the propagation with respect to the chosen ansatz function. From the results presented in Figs. 1 and 2 it is obvious that there is very good agreement with the analytical solutions of the variational equations and the numerical solution of the NLSE.

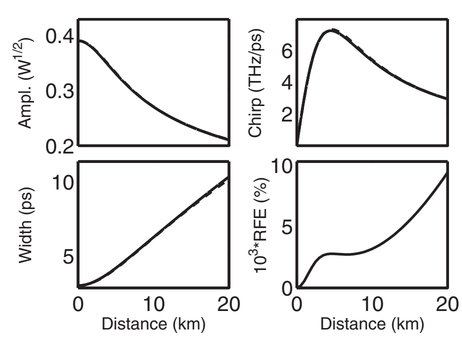


FIG. 1. Evolution of the pulse amplitude, width, and chirp for the propagation of a Gaussian shaped pulse. Solid and dashed curves show the results obtained from the analytical solutions of the variational equations and the numerical solution of the NLSE (1), respectively.

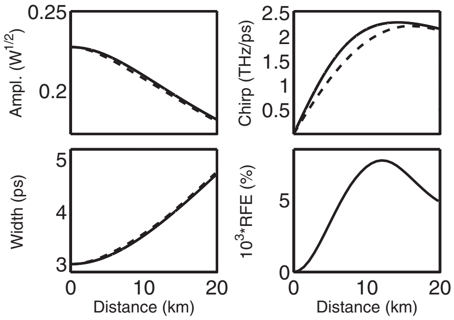


FIG. 2. Evolution of the pulse amplitude, width, and chirp for the propagation of a hyperbolic secant shaped pulse. Solid and dashed curves show the results obtained from the analytical solutions of the variational equations and the numerical solution of the NLSE (1), respectively.

## V. CONCLUSION

To conclude, we have found the direct relation between the pulse energy, width, and chirp for a pulse propagating in a nonlinear fiber with anomalous dispersion, in the limit where the Hamiltonian of the system is zero. Then, using that result, we have found the exact analytical expressions for the amplitude, width, and chirp of the pulse with respect to the fiber parameters and initial pulse width. For the Gaussian and hyperbolic secant ansätze, we have compared the results obtained from the analytical solutions of the variational equations with the results of the direct numerical simulation of the NLSE. As the analytical solutions for the variational equations are obtained for any general ansatz function, the expressions obtained could be used for the pulse with any profile. We believe that the analytical results reported in this work could be utilized not only in the context of optical fibers but also to any system governed by the NLSE.

## ACKNOWLEDGMENTS

The Agence Universitaire pour la Francophonie is gratefully acknowledged for the financial support in the work of A.-B.M. K.N. wishes to thank The Nuffield Foundation for their financial support.

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